Density maximizers of layered permutations

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A permutation is called *layered* if it can be obtained from the identity permutation $(1, 2, \dots, k)$ by splitting it and reverting the ordering within each layer. For instance, the permutation $(13, 12, \dots, 1, 14, 16, 15)$ is layered; it has three layers and the sizes of the layers are $13, 1, 2$. It is known that for every layered permutation π and every positive integer n, the set of all permutations of length n maximising the density of π contains a layered permutation. Furthermore, if π has no layer of size 1, then all density maximisers are layered. Known results also indicate that the layers of size 1 in π play a key role in determining the structure of the layered density maximisers.

We consider a layered permutation π and the density maximisers of π which are layered, and we investigate the question whether the number of their layers can be bounded. We show that the answer is negative if the first layer of π is long and the second layer is of size 1; for instance, π can be chosen as $(13, 12, \dots, 1, 14, 16, 15)$. This disproves a conjecture of Albert, Atkinson, Handley, Holton and Stromquist [Electronic Journal of Combinatorics (2002)] which suggested a positive answer for every π having no consecutive layers of size 1 and the first and last layer of size at least 2. We complement this result by showing that the conjecture is true under additional assumptions concerning the shortness of the first and last layer.